I. FIELD MANIPULATION WITH LORENE

The aim is simply to get used to LORENE library for the definition, manipulation, computation and drawing of scalar and vector fields in spherical coordinates and/or components. For all classes and functions, please look carefully at the documentation at `Lorene/Doc/refguide/index.html`.

- Setup a multi-domain three-dimensional grid. It should contain a nucleus, one or more shells and a compactified external domain. Take it to be symmetric / equatorial plane and not symmetric / $(x, y) \to (-x, -y)$.
- Using coordinate fields (`Coord` objects, members of the mapping), define a *regular* 3D (but symmetric / equatorial plane) scalar field of type `Scalar`.
- After setting the spectral base, draw iso-contours with `des_coupe_...` and profiles with `des_meridian`.
- Compute the radial derivative of the field and compare it to the "analytic" value (*e.g.* using `maxabs` or `diffrelmax`).
- Define a regular vector field in spherical triad, either by setting it first in a Cartesian triad and changing the triad, or as the gradient of a scalar field (covariant derivative / flat metric). Draw the vector field.

II. TEST OF A ROTATING BLACK HOLE METRIC

With tensor calculus tools, it is easy to check whether a given metric is solution of Einstein equations. As an example, the Kerr-Schild metric shall be tested, within the framework of the 3+1 formalism. This metric provides a description of a rotating black hole (*i.e.* vacuum space-time), with a mass M and the angular momentum per unit mass a:

$$g_{\mu\nu} = f_{\mu\nu} + 2H l_\mu l_\nu; \tag{1}$$

where $f_{\mu\nu}$ is the flat metric,

$$H = \frac{M\rho^3}{\rho^4 + a^2 z^2} \tag{2}$$

and

$$l_\mu = \left(1, \frac{\rho x + ay}{\rho^2 + a^2}, \frac{\rho y - ax}{\rho^2 + a^2}, \frac{z}{\rho}\right). \tag{3}$$

Note that the spatial components of l_μ are expressed in a Cartesian triad and ρ is related to the usual radial coordinate r – (x, y, z) being the usual Cartesian coordinates – by the relation:

$$\rho^2 = \frac{1}{2}\left(r^2 - a^2\right) + \sqrt{\frac{1}{4}\left(r^2 - a^2\right)^2 + a^2 z^2}. \tag{4}$$

To test it, the metric should be written in the 3+1 form[1] (using only 3-tensors):

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt); \tag{5}$$

with N being the lapse, β the shift and γ_{ij} the 3-metric. In this case:

$$\begin{aligned} N &= \frac{1}{\sqrt{1+2H}}; \\ \beta_i &= 2H l_i; \\ \gamma_{ij} &= f_{ij} + 2H l_i l_j \end{aligned}$$

[1] latin indices range from 1 to 3 (only spatial components), whereas greek ones range from 0 to 3

One also defines the extrinsic curvature

$$K_{ij} = \frac{1}{2N}\left(\pounds_{\boldsymbol{\beta}}\gamma_{ij} - \frac{\partial}{\partial t}\gamma_{ij}\right) \tag{6}$$

$\pounds_{\boldsymbol{\beta}}\gamma_{ij}$ being the Lie-derivative along the shift of the 3-metric.

The ten Einstein equations write (in vacuum):

- the Hamiltonian constraint equation:

$$R + K^2 - K_{ij}K^{ij} = 0, \tag{7}$$

- the three momentum constraint equations

$$D_j K_i^{\ j} - D_i K = 0, \tag{8}$$

- and the six dynamical evolution equations

$$\frac{\partial}{\partial t}K_{ij} - \pounds_{\boldsymbol{\beta}}K_{ij} = -D_i D_j N + N\big[R_{ij} - 2K_{ik}K^k_{\ j} + KK_{ij}\big]. \tag{9}$$

  D_i is the covariant derivative / γ_{ij}, K the trace of K_{ij}, R_{ij} and R the Ricci tensor and scalar associated with this 3-metric.

III. SUGGESTED STEPS

- Define a grid (symmetric / $(x, y) \to (-x, -y)$ transform), with at least 4 points in φ to be able to rotate from Cartesian triad to the spherical one. Either this grid is without the nucleus, to excise the black hole singularity, or all fields should be set to 0 or 1 in the nucleus to discard the divergence near the centre. Define a mapping on that grid.
- Setup the Kerr-Schild metric described above, with the lapse, shift and the 3-metric.
- Verify that the 1+3+6 equations above are satisfied, using the appropriate methods of classes `Tensor`, `Vector` and `Metric`. Be very careful with the `dzpuis` flag!